A Formula for the Number of Spanning Trees in Quasi-threshold Graphs

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Abstract. In this paper we consider the problem of computing the number of spanning trees in the class of quasi-threshold graphs, or QT-graphs for short. We show that such a graph admits important structural and algorithmic properties among which a unique tree representation, up to isomorphism, called cent-tree. Based on the properties of the cent-tree of a QT-graph G we derive a formula which gives the number of spanning trees of the graph G; the proof is based on the Kirchhoff matrix tree theorem. Our result generalizes and extends previous results regarding the number of spanning trees of QT-graphs [16].

1 Introduction

We consider finite undirected graphs with no loops nor multiple edges. Let G be such a graph on n vertices. A *spanning tree* of G is an acyclic $(n-1)$-edge subgraph. The problem of calculating the number of spanning trees on the graph G is an important, well-studied problem in graph theory. Deriving formulas for different types of graphs can prove to be helpful in identifying those graphs that contain the maximum number of spanning trees. Such an investigation has practical consequences related to network reliability [13, 20].

Thus, for both theoretical and practical purposes, we are interested in deriving formulas for the number of spanning trees of classes of graphs. Many cases have been examined depending on the choice of G. It has been studied when G is a labelled molecular graph [2], when G is a circulant graph [25], when G is a complete multipartite graph [23], when G is a cubic cycle and quadruple cycle graph [24], when G is a threshold graph [7] and so on (see Berge [1] for an exposition of the main results; also see [4, 11, 18, 16, 19, 21–23]).

The purpose of this paper is to study the problem of finding the number of spanning trees in the class of *quasi-threshold* graphs. We point out that a graph G is called quasi-threshold graph if it contains no induced subgraph isomorphic to P_4 or C_4 [6, 15, 16]. A quasi-threshold graph G has a unique tree representation $T_c(G)$ called cent-tree. Our proofs are based on a classic result known as the *Kirchhoff Matrix Tree Theorem* [8], which expresses the number of spanning trees of a graph G as a function of the determinant of a matrix (Kirchhoff Matrix) that can be easily construct from the adjacency relation (adjacency matrix, adjacency lists, ect) of the graph G. Calculating the determinant of the Kirchhoff Matrix seems to be a promising approach for computing the number of spanning trees of families of graphs (see [1, 4, 5, 18, 23]). In our case, we compute the number of spanning trees of a quasi-threshold G, using standard techniques from linear algebra and matrix theory. Our ideas and techniques will be formalized and further clarified in the sequel.

The paper is organized as follows. In Section 2 we establish the notation and related terminology and we present background results. In particular, we show structural properties for the quasi-threshold graphs and define a unique tree representation on such a graph. In Sections 3 we present a formula for the number of spanning trees of a quasi-threshold graph. Finally, in Section 4 we conclude the paper and discuss possible future extensions.

2 Definitions and Background Results

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The neighborhood $N(x)$ of a vertex $x \in V(G)$ is the set of all the vertices of G which are adjacent to x. The closed neighborhood of x is defined as $N[x] := \{x\} \cup N(x)$ [8].

The subgraph of a graph G induced by a subset S of the vertex set $V(G)$ is denoted by $G[S]$. For a vertex subset S of G, we define $G - S := G[V(G) - S]$.

2.1 Quasi-threshold Graphs

A graph G is called a *quasi-threshold* graph, or *QT-graph* for short, if G has no induced subgraph isomorphic to P_4 or C_4 [6, 15, 16]. We next provide characterizations and structural properties of QT-graphs and show that such a graph has a unique tree representation. The following lemma follows immediately from the fact that for every subset $S \subset V(G)$ and for a vertex $u \in S$, we have $N_{G[S]}[u] = N[u] \cap S$ and that $G[V(G) - S]$ is an induced subgraph.

Lemma 2.1. ([10]): *If G is a QT-graph, then for every subset $S \subset V(G)$, both $G[S]$ and $G[V(G) - S]$ are also QT-graphs.*

The following theorem provides important properties for the class of QT-graphs. For convenience, we define

$$cent(G) = \{x \in V(G) \mid N[x] = V(G)\}.$$

Theorem 2.1. ([10, 15]): *The following three statements hold.*

(i) A graph G is a QT-graph if and only if every connected induced subgraph $G[S], S \subseteq V(G)$, satisfies $cent(G[S]) \neq \emptyset$.
(ii) A graph G is a QT-graph if and only if $G[V(G) - cent(G)]$ is a QT-graph.
(iii) Let G be a connected QT-graph. If $V(G) - cent(G) \neq \emptyset$, then $G[V(G) - cent(G)]$ contains at least two connected components.

Let G be a connected QT-graph. Then $V_1 := cent(G)$ is not an empty set by Theorem 2.1. Put $G_1 := G$, and $G[V(G) - V_1] = G_2 \cup G_3 \cup \ldots \cup G_r$, where each G_i is a connected component of the graph $G[V(G) - V_1]$ and $r \geq 3$. Then since each G_i is an induced subgraph of G, G_i is also a QT-graph, and so let $V_i := cent(G_i) \neq \emptyset$ for $2 \leq i \leq r$. Since each connected component of $G_i[V(G_i) - cent(G_i)]$ is also a QT-graph, we can continue this procedure until we get an empty graph. Then we finally obtain the following partition of $V(G)$.

$$V(G) = V_1 + V_2 + \ldots + V_k, \quad \text{where } V_i = cent(G_i).$$

Moreover we can define a partial order $\preceq$ on $\{V_1, V_2, \ldots, V_k\}$ as follows:

$$V_i \preceq V_j \ \text{ if } \ V_i = cent(G_i) \ \text{ and } \ V_j \subseteq V(G_i).$$

It is easy to see that the above partition of $V(G)$ possesses the following properties.

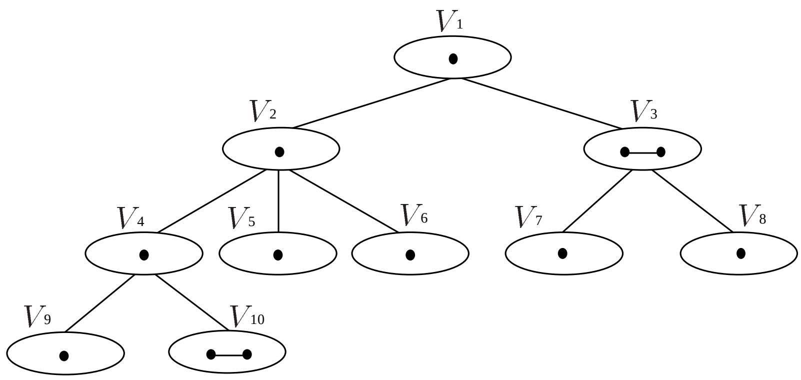

Fig. 1: A cent-tree $T_c(Q)$ of a QT-graph on 12 vertices.

Theorem 2.2. ([10, 15]): *Let G be a connected QT-graph, and let $V(G) = V_1 + V_2 + \ldots + V_k$ be the partition defined above; in particular, $V_1 := cent(G)$. Then this partition and the partially ordered set $(\{V_i\}, \preceq)$ have the following properties:*

(P1) If $V_i \preceq V_j$, then every vertex of V_i and every vertex of V_j are joined by an edge of G.

(P2) For every V_j, $cent(G[\{\cup V_i \mid V_i \preceq V_j\}]) = V_j$.

(P3) For every two V_s and V_t such that $V_s \preceq V_t$, $G[\{\cup V_i \mid V_s \preceq V_i \preceq V_t\}]$ is a complete graph. Moreover, for every maximal element V_t of $(\{V_i\}, \preceq)$, $G[\{\cup V_i \mid V_1 \preceq V_i \preceq V_t\}]$ is a maximal complete subgraph of G.

The results of Theorem 2.2 provide structural properties for the class of QT-graphs. We shall refer to the structure that meets the properties of Theorem 2.2 as *cent-tree* of the graph G and denote it by $T_c(G)$. The cent-tree is a rooted tree with root V_1; every node V_i of the tree $T_c(G)$ is either a leaf or has at least two children. Moreover, $V_s \preceq V_t$ if and only if V_s is an ancestor of V_t in $T_c(G)$. Here, we define $\text{ch}(V_i)$ to be the set which contains the children of the node $V_i \in T_c(G)$; we shall use $\text{ch}(i)$ to denote $\text{ch}(V_i)$, $1 \leq i \leq k$.

In Figure 1 we show a cent-tree of a QT-graph on 12 vertices. Nodes V_3 and V_{10} contain two vertices, while all the other contain one vertex; $\text{ch}(V_3) = \{V_7, V_8\}$ and $\text{ch}(V_{10}) = \emptyset$. Notice that the degree of a vertex in node V_3 is 4.

2.2 Kirchhoff Matrix

For an $n \times n$ matrix A, the ijth *minor* is the determinant of the $(n-1) \times (n-1)$ matrix M_{ij} obtained from A deleting row i and column j. The ith *cofactor* denoted A_i equals $det(M_{ii})$.

Let G be a graph on n vertices. Then the *Kirchhoff matrix* K for the graph G has

$$k_{i,j} = \begin{cases} d_i & \text{if } i = j, \\ -1 & \text{if } i \neq j \text{ and } (i,j) \in E, \\ 0 & \text{otherwise,} \end{cases}$$

elements, where d_i is the number of edges incident to vertex v_i in the graph G. The Kirhhoff Matrix Tree Theorem is one of the most famous results in graph theory. It provides a formula for the number of spanning trees of a graph G, in terms of the cofactors of its Kirhhoff Matrix.

Theorem 2.3. (Kirchhoff Matrix Tree Theorem [8]): *For any graph G with K defined as above, the cofactors of K have the same value, and this value equals the number of spanning trees of G.*

3 The Number of Spanning Trees

In this section we derive a formula for the number of spanning trees of a QT-graph G; hereafter, $\tau(G)$ denotes the number of spanning trees of G.

Let G be a QT-graph on n vertices and let $V_1, V_2, \ldots, V_k$ be the nodes of its cent-tree $T_c(G)$ containing $n_1, n_2, \ldots, n_k$ vertices, respectively; that is, $n = n_1 + n_2 + \ldots + n_k$. We let d_i denote the degree of an arbitrary vertex of the node V_i. Recall that all the vertices $u \in V(G)$ of a node V_i have the same degree.

Let $V_1, V_2, \ldots, V_k$ be the nodes of the cent-tree $T_c(G)$ of a QT-graph on n vertices. We denote L_i the set which contains the nodes of the ith level of $T_c(G)$, $1 \leq i \leq h$; that is,

$$\begin{aligned} L_0 &= \{V_1\}, \\ L_1 &= \{V_2, V_3, \ldots, V_r\}, \\ &\vdots \\ L_{h-1} &= \{V_s, V_{s+1}, \ldots, V_{\ell-1}\}, \\ L_h &= \{V_\ell, V_{\ell+1}, \ldots, V_k\}. \end{aligned}$$

3.1 The Formula

We next form the Kirhhoff matrix K for the QT-graph G based on the structure of the cent-tree $T_c(G)$.

Let G be a QT-graph on n vertices and let $V_1, \ldots, V_s, \ldots, V_{\ell-1}, V_\ell, \ldots, V_k$ be the nodes of its cent-tree $T_c(G)$. Then, we label the vertices of the graph G from 1 to n as follows: First, we label the vertices in node V_k from 1 to n_k; next, we label the vertices in V_{k-1} from $n_k + 1$ to $n_k + n_{k-1}$; finally, we label the vertices in node V_1.

Then, we construct the matrix K of the QT-graph G, using the above defined labelling of the vertices of G, and we focus on the determinant of the matrix K_{nn} obtained from K by deleting its last row and its last column. It is easy to see that K_{nn} is an $(n-1) \times (n-1)$ matrix and has the following form:

$$K_{nn} = \begin{bmatrix} M_k & & & & & & & \\ & \ddots & & & & & [-1]_{ji} & \\ & & M_\ell & & & & & \\ & & & M_{\ell-1} & & & & \\ & & & & \ddots & & & \\ & & & & & M_s & & \\ & [-1]_{ij} & & & & & \ddots & \\ & & & & & & & M_1 \end{bmatrix}, \tag{1}$$

where M_i is an $n_i \times n_i$ submatrix of the form

$$M_i = \begin{bmatrix} d_i & -1 & \cdots & -1 \\ -1 & d_i & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & d_i \end{bmatrix},$$

for $2 \le i \le k$ and M_1 is an $(n_1 - 1) \times (n_1 - 1)$ submatrix of the same form. All the diagonal positions of the matrix M_i have the same value d_i which equals the degree of an arbitrary vertex in node $V_i \in T_c(G)$; the entries $[-1]_{ij}$ and $[-1]_{ji}$ of the off-diagonal positions (i, j) and (j, i) correspond to $n_i \times n_j$ and $n_j \times n_i$ submatrices with all their elements -1 if node V_j is a descendant of node V_i in $T_c(G)$ and all their elements 0 otherwise, $2 \le i,\ j \le k$; the entries $[-1]_{1,j}$ and $[-1]_{j,1}$ of the off-diagonal positions $(1, j)$ and $(j, 1)$ correspond to $(n_1 - 1) \times n_j$ and $n_j \times (n_1 - 1)$ submatrices with all their elements -1 since every node V_j is descendant of the root V_1 of the cent-tree $T_c(G)$.

In order to compute the determinant of the matrix K_{nn} we first simplify the determinants of the matrices M_i, $2 \le i \le k$, and we obtain:

$$\det(M_i) = \begin{vmatrix} d_i+1 & & & \\ & d_i+1 & & \\ & & \ddots & \\ -1 & -1 & & d_i+1-n_i \end{vmatrix} = (d_i+1)^{n_i-1}(d_i-(n_i-1)),$$

and

$$det(M_1) = (d_1+1)^{n_1-2}(d_1-(n_1-2)).$$

It now suffices to substitute the above values in the determinant of the matrix K_{nn}. We point out that after simplifying the determinants of matrices M_i only the diagonal and the last row of each matrix M_i have non-zero's entries. Thus, we have:

$$\det(K_{nn}) = (n_1-1)(d_1+1)^{n_1-2} \cdot \prod_{i=2}^{k} n_i(d_i+1)^{n_i-1} \cdot \det(B_{nn}), \tag{2}$$

where

$$B_{nn} = \begin{bmatrix} \sigma_k & & & & & & & \\ & \ddots & & & & (-1)_{ji} & & \\ & & \sigma_\ell & & & & & \\ & & & \sigma_{\ell-1} & & & & \\ & & & & \ddots & & & \\ & & & & & \sigma_s & & \\ & (-1)_{ij} & & & & & \ddots & \\ & & & & & & & \sigma_1 \end{bmatrix} \tag{3}$$

is a $k \times k$ matrix with diagonal elements $\sigma_i = \frac{d_i-(n_i-1)}{n_i}$ for $2 \leq i \leq k$, and $\sigma_1 = \frac{d_1-(n_1-2)}{n_1-1}$; the entry $(-1)_{ij}$ of the off-diagonal position (i,j) is -1 if node V_j is a descendant of node V_i in $T_c(G)$ and 0 otherwise.

The idea now is to compute the determinant of the $k \times k$ matrix B_{nn}. To this end, we first apply the following operations to each row $i = 1, 2, \ldots, k$ of the matrix B_{nn}:

- we find the minimum index j such that $i < j \leq k$ and $B_{nn}[i,j] \neq 0$, and then
- we multiply the jth column by -1 and add it to the ℓth column if $B_{nn}[i,\ell] = B_{nn}[i,j]$ and $j+1 \leq \ell \leq k$.

Next, we apply similar operations to each column $j = 1, 2, \ldots, k$ of the matrix B_{nn}:

- we find the minimum index i such that $1 \leq j < i$ and $B_{nn}[i,j] \neq 0$, and then
- we multiply the ith row by -1 and add it to to the ℓth row if $B_{nn}[\ell,j] = B_{nn}[i,j]$ and $i+1 \leq \ell \leq k$.

Thus, we obtain:

$$\det(B_{nn}) = \det(A_{nn}), \tag{4}$$

where

$$A_{nn} = \begin{bmatrix} a_k & & & & & & & \\ & \ddots & & & & (\gamma)_{ji} & & \\ & & a_\ell & & & & & \\ & & & a_{\ell-1} & & & & \\ & & & & \ddots & & & \\ & & & & & a_s & & \\ & (\gamma)_{ij} & & & & & \ddots & \\ & & & & & & & a_1 \end{bmatrix}, \tag{5}$$

is a $k \times k$ matrix and

$$a_i = \begin{cases} \sigma_i & \text{if } V_i \text{ is a leaf of } T_c(G), \\ \sigma_i + \sum\limits_{\substack{j \in ch(i) \\ j \text{not a leaf}}} (\sigma_j + 2) & \text{otherwise,} \end{cases} \tag{6}$$

and

$$(\gamma)_{ij} = \begin{cases} -1 & \text{if } V_j \in ch(V_i) \text{ and } V_j \text{ is a leaf of } T_c(G), \\ -(\sigma_i + 1) & \text{if } V_j \in ch(V_i) \text{ and } V_j \text{ is not a leaf of } T_c(G), \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Recall that

$$\sigma_i = \begin{cases} \dfrac{d_i - (n_i - 2)}{n_i - 1} & \text{if } i = 1 \\ \dfrac{d_i - (n_i - 1)}{n_i} & \text{otherwise.} \end{cases} \tag{8}$$

In the case where each node of the cent-tree $T_c(G)$ contains a single vertex, we have $\sigma_i = d_i$ for every $i = 2, 3, \ldots, k$; note that $i \geq 2$, since we delete the last row and column of the matrix K.

We next define the following function ϕ on the nodes on the cent-tree of a QT-graph G:

$$\phi(i) = \begin{cases} a_i & \text{if } V_i \text{ is a leaf of } T_c(G), \\ a_i - \displaystyle\sum_{j \in ch(i)} \frac{((\gamma)_{ij})^2}{\phi(j)} & \text{otherwise,} \end{cases} \tag{9}$$

where a_i and $(\gamma)_{ij}$ are defined in Eq. (6) and Eq. (7), respectively. We call the function $\phi(i)$ *cent-function* of the node V_i; hereafter, we use ϕ_i to denote $\phi(i)$, $1 \leq i \leq k$.

Lemma 3.1. *Let $V_1, V_2, \ldots, V_k$ be the nodes of the cent-tree $T_c(G)$ of a QT-graph G and let $\phi(i)$ be the cent-function of V_i, $1 \leq i \leq k$. Then,*

$$\prod_{i=1}^{k} \phi(i) = det(A_{nn}),$$

where A_{nn} is the $k \times k$ matrix defined in Eq. (5).

Proof. In order to compute the determinant $\det(A_{nn})$, we start by multiplying each column i, $1 \leq i \leq \ell$, of the matrix A_{nn} by $-(\gamma)_{ij}/a_i$ and adding it to the column j if $(\gamma)_{ij} \neq 0$ $(i < j \leq k)$. This, makes all the strictly upper-diagonal entries $(\gamma)_{ij}$, that is, $i < j$, into zeros. Now expand in terms of the $1, 2, \ldots, \ell$ rows, getting

$$\det(A_{nn}) = \prod_{i=1}^{\ell} \phi_i \cdot \begin{vmatrix} f_{\ell-1} & & & & \\ & \ddots & & & (\gamma)_{ji} \\ & & f_s & & \\ & (\gamma)_{ij} & & \ddots & \\ & & & & f_1 \end{vmatrix} = \prod_{i=1}^{\ell} \phi_i \cdot det(D_{nn}),$$

where

$\phi_i = a_i$, for $1 \leq i \leq \ell$, since the nodes $1, 2, \ldots, \ell$ are leaves of $T_c(G)$, and

$$f_t = a_t - \sum_{\substack{i \in ch(t) \\ 1 \leq i \leq \ell}} \frac{((\gamma)_{it})^2}{\phi_i}, \quad \text{for } \ell + 1 \leq t \leq k.$$

We observe that the $(k-\ell)\times(k-\ell)$ matrix D_{nn} has a structure similar to that of the initial matrix A_{nn}; see Eq. 5. Thus, for the computation of its determinant $\det(D_{nn})$, we follow a similar simplification; that is, we start by multiplying each column i, $1 \le i \le s$, of the matrix D_{nn} by $-(\gamma)_{ij}/f_j$ and adding it to the column j if $(\gamma)_{ij} \neq 0$, for $s < j \le k$. Thus, continuing in the same fashion we can finally show that

$$\det(D_{nn}) = \prod_{i=1}^{k} \phi_i,$$

where ϕ_i is the cent-function of the node $V_i \in T_c(G)$ and k is the number of nodes of the cent-tree $T_c(G)$. ∎

Based on Eq. (2), (4) and Lemma 3.1 we can obtain a formula for the number of spanning trees $\tau(G)$ of a quasi-threshold graph G. Thus, we present the following result.

Theorem 3.1. *Let G be a quasi-threshold graph on n vertices and let $V_1, V_2, \ldots, V_k$ be the nodes of the cent-tree $T_c(G)$ rooted at node V_1. Then,*

$$\tau(G) = \frac{n_1 - 1}{n_1(d_1+1)} \cdot \prod_{i=1}^{k} n_i(d_i+1)^{n_i-1} \cdot \phi_i,$$

where n_i is the number of vertices of the node V_i, d_i is the degree of an arbitrary vertex of V_i and ϕ_i is the cent-function of the node V_i, $1 \le i \le k$.

Remark 3.1. Based on the above formula, we propose a linear-time algorithm for determining the number of spanning trees of a QT-graph; it works as follows: First it computes the cent-tree $T_c(G)$ of the quasi-threshold graph; let $V_1, V_2, \ldots, V_k$ be the nodes of the cent-tree $T_c(G)$. Then, it computes the cent-function ϕ_i of each node $V_i \in T_c(G)$, $1 \le i \le k$, and, finally, it computes the number of spanning trees of the quasi-threshold graph based on the result presented in Theorem 3.1.

We point out that the number of spanning trees of a QT-graph G on n vertices and m edges can be computed in $O(n+m)$ time. The construction of a cent-tree $T_c(G)$ takes $O(n+m)$ time using a DFS traversal on the input QT-graph. Moreover, the computation of all the cent-functions ϕ_i, $1 \le i \le k$, can be performed in $O(n)$ time, since the number of the nodes of the cent-tree $T_c(G)$ is $k \le n$. Thus, the proposed algorithm runs in $O(n+m)$ time.

The time complexity is measured according to the uniform cost criterion. Under this criterion each instruction requires one unit of time and each register requires one unit of space. Despite the fact that the arithmetic operations involve arbitrarily large integers, we count each operation as a single step (the number of spanning trees of a graph G on n vertices can be at most n^{n-2}; the complete graph K_n has n^{n-2} spanning trees). □

4 Concluding Remarks

In this paper we derived a formula for the number of spanning trees of a quasi-threshold graph using the Kirchhoff Matrix Tree Theorem and taking advantage of the structural properties of the cent-tree of a quasi-threshold graph.

Another class of perfect graphs, called cographs, are precisely the graphs containing no chordless path on four vertices (termed a P_4). In [17], a linear-time algorithm is given for computing the number of spanning trees of cographs based on a unique rooted tree, called the cotree. Thus, an interesting question is whether we can derive a formula for the number of spanning trees in the class of cographs.

More general classes of perfect graphs, such as the classes of P_4-reducible and P_4-sparse graphs, also admit unique tree representations. Thus, it is reasonable to ask whether the structural properties of these tree representations are helpful to derive formulas regarding the number of spanning trees of the corresponding graphs.

It has been shown that a permutation graph $G[\pi]$, a well-known class of perfect graphs, can be transform into a directed acyclic graph and, then, into a rooted tree by exploiting the inversion relation on the elements of the permutation π [14]. Based on these results, one can work towards the investigation whether the class of permutation graphs $G[\pi]$ belong to the family of graphs that admit formulas for the number of their spanning trees.

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