Using Neural Networks to Simulate the Logistic Map: a Theoretical Approach

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Abstract. The aim of this research is to establish a theoretical framework for the neural modeling of chaotic attractors. The attractor paradigm, studied in this paper, is the logistic equation, which is modeled via neural networks in the convergence, periodic, and chaotic regions. In the first step, it is proven that under some conditions the function simulated by the neural model is actually the logistic map with a different value of the lambda parameter than the theoretical one. Then, a two dimensional system is defined and studied, that allows the generation of the theoretical time series and the associated simulation error. The fixed points of period p=1 and p=2 are identified, and studied with respect to their stability. For higher period values, two theorems associated with the periodicity of the simulation error are postulated and proven. Finally, the minimum simulation error value is calculated through analytical methods and the chaotic nature of the system with respect to Lyapunov exponents is described.

1 Introduction

Neural modelling is a technique that is broadly used for the construction of models of various types of dynamical systems. In this study, the system under consideration is the logistic map $y(x) = \lambda_{thr}x(1-x)$, which is modeled though the construction and training of three layered feed forward neural networks, trained with the back propagation algorithm [1,2,3,4]. The neural models are then used to generate the curve of the logistic equation, as well as the time series associated with various values of the λ parameter. According to the value of this parameter, the experimental time series produced by the network is converging, periodic, or chaotic. This time series, is generated via an iterative equation of the form $x_{n+1} = F(x_n)$, where, F is the function that is simulated by the neural network.

To map the neural function F to the logistic map function, we make the assumption that it can be expressed as a second degree polynomial in the form $y(x) = \alpha x^2 + \beta x + \gamma$ where the α, β and γ coefficients can be calculated through experimentation. Then we evaluate the Lagrange polynomial [5] that passes from three random experimental points produced by the network. It is proven that the coefficients of this polynomial are almost the same for any combination of the experimental points used in groups of three points. Furthermore, if the simulation accuracy is very good, it is proven that $\gamma \to 0$, and $\beta = -\alpha$. Therefore, if we set $\gamma = 0$ and $-\alpha = \beta = \lambda_{exp}$, the Lagrange polynomial can be written in the form $y(x) = \lambda_{exp}x(1-x)$. This means that the function produced by the neural model is actually the logistic equation with a parameter value

λ_{exp} different than the value λ_{thr} of the theoretical model. It can be proven that the λ parameter value of the logistic equation can be evaluated by means of the time series values $\{x_1, x_2, \ldots, x_n\}$ through the equation [6]:

$$\lambda = (\sum_{i=1}^{N} x_i) \Big/ (\sum_{i=0}^{N} x_i(1 - x_i)) \tag{1}$$

Therefore, the theoretical and the experimental time series points can be used to evaluate the values λ_{thr} and λ_{exp} of the system parameter.

2 Setting up the Theoretical Simulation Model

In order to setup the theoretical model for the neural based simulation of the logistic map, we can combine the equations $x_{n+1}^{thr} = \lambda_{thr} x_n^{thr}(1 - x_n^{thr})$ and $x_{n+1}^{exp} = \lambda_{exp} x_n^{exp}(1 - x_n^{exp})$ that describe the theoretical and the experimental time series, getting the simulation error time series defining equation

$$e_{n+1} = \lambda_{exp}(e_n)^2 + \lambda_{exp} e_n (1 - 2x_n^{thr}) + (\lambda_{thr} - \lambda_{exp}) x_n^{thr} (1 - x_n^{thr}) \tag{2}$$

If this equation is combined with the equation of the original logistic map, the two dimensional discrete time dynamical system:

$$\begin{aligned} e_{n+1} &= f_1(e_n, x_n^{thr}) = \lambda_{exp}(e_n)^2 + \lambda_{exp} e_n (1-2x_n^{thr}) + (\lambda_{thr} - \lambda_{exp}) x_n^{thr} (1-x_n^{thr}) \\ x_{n+1}^{thr} &= f_2(e_n, x_n^{thr}) = \lambda_{thr} x_n^{thr} (1 - x_n^{thr}) \end{aligned}$$

is defined. The state variables of this system are the theoretical time series sample x_n^{thr} and the simulation error e_n between this sample and the corresponding experimental sample produced by the neural model. The main features of this system are presented in the next sections.

2.1 Fixed Points with Period $p = 1$

According to the basic theory of the discrete time dynamical systems [7], the fixed points of the system under consideration must satisfy the equation $\boldsymbol{p} = \boldsymbol{F}(\boldsymbol{p})$ which can be expressed as:

$$\begin{aligned} e &= \lambda_{exp} e^2 + \lambda_{exp}(1 - 2x)e + (\lambda_{thr} - \lambda_{exp}) x (1 - x) \\ x &= \lambda_{thr} x (1 - x) \end{aligned}$$

The solution of the above system leads to four different fixed points with period $p = 1$ with coordinates:

$$\begin{aligned} \boldsymbol{p}_1 &= (e_1, x_1) = (0, 0) \\ \boldsymbol{p}_2 &= (e_2, x_1) = (-\frac{(\lambda_{exp} - 1)}{\lambda_{exp}}, 0) \end{aligned}$$

$$\boldsymbol{p}_3 = (e_1, x_2) = (\frac{(\lambda_{thr} - 1)}{\lambda_{thr}}, \frac{(\lambda_{thr} - 1)}{\lambda_{thr}})$$
$$\boldsymbol{p}_4 = (e_2, x_1) = (\frac{\lambda_{thr} - \lambda_{exp}}{\lambda_{thr}\lambda_{exp}}, \frac{(\lambda_{thr} - 1)}{\lambda_{thr}})$$

The positions of these points in the system's phase space, define two diagonal lines formed by the points $(\boldsymbol{p}_1, \boldsymbol{p}_3)$ and $(\boldsymbol{p}_2, \boldsymbol{p}_4)$ respectively. These lines cross the horizontal axis in the coordinates $x_1^{exp} = 0$ and $x_2^{exp} = (\lambda_{exp} - 1)/\lambda_{exp}$, that are equal to the fixed points with period $p = 1$ associated with the logistic map neural model.

Stability of the fixed points: The stability of the fixed points with period $p = 1$ depends on the measure of the eigenvalues of the Jacobian matrix calculated on those points. By extracting the analytic form of the partial derivatives of the functions $f_1(e_n, x_n^{thr})$ and $f_2(e_n, x_n^{thr})$ and by substituting the coordinates of the fixed points as they have been calculated in the previous section, the Jacobian matrices $J(\boldsymbol{p}_i)$ $(i = 1, 2, 3, 4)$ are found to have the form:

$$J(\boldsymbol{p}_1) = \begin{pmatrix} \lambda_{exp} & \lambda_{thr} - \lambda_{exp} \\ 0 & \lambda_{thr} \end{pmatrix} \quad J(\boldsymbol{p}_2) = \begin{pmatrix} 2 - \lambda_{exp} & \lambda_{thr} + \lambda_{exp} - 2 \\ 0 & \lambda_{thr} \end{pmatrix}$$

$$J(\boldsymbol{p}_3) = \begin{pmatrix} \lambda_{exp} & 2 - \lambda_{thr} - \lambda_{exp} \\ 0 & 2 - \lambda_{thr} \end{pmatrix} \quad J(\boldsymbol{p}_4) = \begin{pmatrix} 2 - \lambda_{exp} & \lambda_{exp} - \lambda_{thr} \\ 0 & 2 - \lambda_{thr} \end{pmatrix}$$

The type of stability of the fixed points depends on the eigenvalues of the matrix $J(\boldsymbol{p}_i)$ $(i = 1, 2, 3, 4)$. These eigenvalues are evaluated as the roots of the polynomial

$$\mu(\boldsymbol{p}_i)^2 - Trace(J(\boldsymbol{p}_i))\mu(\boldsymbol{p}_i) + Det(J(\boldsymbol{p}_i)) = 0 \tag{3}$$

and they are presented in Table (1) for the fixed points $\boldsymbol{p}_i$ $(i = 1, 2, 3, 4)$.

Table 1. Eigenvalues μ_1 and μ_2 of the period-1 fixed points $\boldsymbol{p}_i$ $(i = 1, 2, 3, 4)$

Eigenvalue $\mu_1(\boldsymbol{p}_i)$	Eigenvalue $\mu_2(\boldsymbol{p}_i)$
$\mu_1(\boldsymbol{p}_1) = \lambda_{exp}$	$\mu_2(\boldsymbol{p}_1) = \lambda_{thr}$
$\mu_1(\boldsymbol{p}_2) = 2 - \lambda_{exp}$	$\mu_2(\boldsymbol{p}_1) = \lambda_{thr}$
$\mu_1(\boldsymbol{p}_3) = \lambda_{exp}$	$\mu_2(\boldsymbol{p}_1) = 2 - \lambda_{thr}$
$\mu_1(\boldsymbol{p}_4) = 2 - \lambda_{exp}$	$\mu_2(\boldsymbol{p}_1) = 2 - \lambda_{thr}$

According to the basic theory of dynamical systems there are four types of fixed points with respect to their stability : (a) stable sink points, (b) unstable source points, (c) type I saddle points and (d) type II saddle points. Regarding the eigenvalues of those points, they must satisfy the inequalities $|\mu_1(\boldsymbol{p}_i)| < 1$ and $|\mu_2(\boldsymbol{p}_i)| < 1$ for the sink points, $|\mu_1(\boldsymbol{p}_i)| > 1$ and $|\mu_2(\boldsymbol{p}_i)| > 1$ for the source points, $|\mu_1(\boldsymbol{p}_i)| < 1$ and $|\mu_2(\boldsymbol{p}_i)| > 1$ for the type I saddle points, and $|\mu_1(\boldsymbol{p}_i)| > 1$ and $|\mu_2(\boldsymbol{p}_i)| < 1$, for the type II saddle points. By using the eigenvalues of the four fixed points as they are shown in table 1, the stability regions of those points form the stability diagram of figure 1. The small square regions of this diagram pointed by the arrows, have a side size length equal to $|\Delta\lambda| = |\lambda_{thr} - \lambda_{exp}|$.

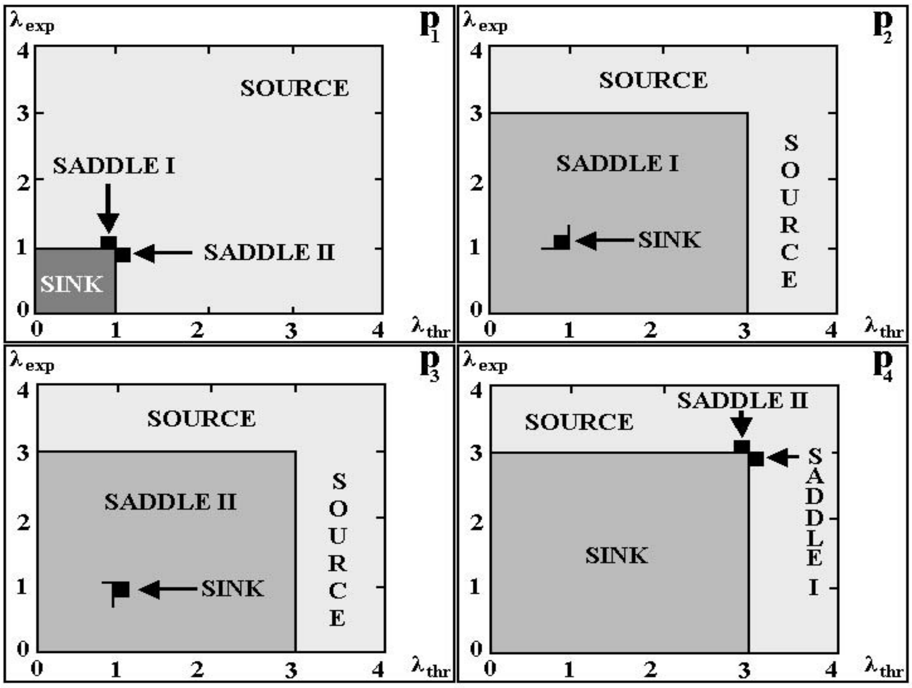

Fig. 1. Stability Diagram of the four fixed points with period $p = 1$ in the diagram $(\lambda_{thr}, \lambda_{exp})$

2.2 Fixed Point with Period $p = 2$

The fixed points with period $p = 2$, satisfy the equation $\boldsymbol{p} = \boldsymbol{F}^2(\boldsymbol{p})$, where $\boldsymbol{p} = (e, x)$, and $\boldsymbol{F} = (f1, f2)$. This relation can also be expressed in the form $e = h(e, x) = f_1(f_1(e, x), f_2(e, x))$ and $x = q(e, x) = f_2(f_1(e, x), f_2(e, x))$ where

$h(e_n, x_n^{thr}) = \lambda_{exp}^3 e_n^4 + 2\lambda_{exp}^3(1 - 2x_n^{thr})e_n^3 + \lambda_{exp}^2(6\lambda_{exp}(x_n^{thr})^2 - 6\lambda_{exp}x_n^{thr} + \lambda_{exp} + 1)e_n^2 - \lambda_{exp}^2(4\lambda_{exp}(x_n^{thr})^3 - 6\lambda_{exp}(x_n^{thr})^2 + 2(\lambda_e xp + 1)x_n^{thr} - 1)e_n + (\lambda_{thr}^2 - \lambda_{exp}^2)x_n^{thr} + (\lambda_{exp}^2(1 + \lambda_{exp}) - \lambda_{thr}^2(1 + \lambda_{thr}))(x_n^{thr})^2 - 2(\lambda_{thr}^3 - \lambda_{exp}^3)(x_n^{thr})^3 + (\lambda_{exp}^3 - \lambda_{thr}^3)(x_n^{thr})^4$ and

$q(e_n, x_n^{thr}) = -\lambda_{exp}^3(x_n^{thr})^4 + 2\lambda_{thr}^3(x_n^{thr})^3 - \lambda_{thr}^2(1 + \lambda_{thr})(x_n^{thr})^2 + \lambda_{thr}^2 x_n^{thr}$

The solution of these equations leads to the evaluation of sixteen roots in the form $\boldsymbol{p}_i = (e_i, x_i)$ $(i = 1, 2, \ldots, 16)$. The most important of these roots are the points $\boldsymbol{p}_1$ and

$\boldsymbol{p}_2$ with coordinates:

$$e_1 = (\frac{1}{2\lambda_{thr}\lambda_{exp}}(\lambda_{exp}(1+\sqrt{\lambda_{thr}^2-2\lambda_{thr}-3})-\lambda_{thr}(1+\sqrt{\lambda_{exp}^2-2\lambda_{exp}-3}))$$
$$x_1 = \frac{1}{2\lambda_{thr}}(1+\lambda_{thr}+\sqrt{\lambda_{thr}^2-2\lambda_{thr}-3}))$$

and

$$e_2 = (\frac{1}{2\lambda_{thr}\lambda_{exp}}(\lambda_{exp}(1-\sqrt{\lambda_{thr}^2-2\lambda_{thr}-3})-\lambda_{thr}(1-\sqrt{\lambda_{exp}^2-2\lambda_{exp}-3}))$$
$$x_2 = \frac{1}{2\lambda_{thr}}(1+\lambda_{thr}-\sqrt{\lambda_{thr}^2-2\lambda_{thr}-3}))$$

respectively. It can be proven that the sixteen points with period $p = 2$ are arranged in four diagonal lines that cross the horizontal axis in the coordinates

$$x_1^{exp} = 0 \quad x_2^{exp} = \frac{\lambda_{exp}-1}{\lambda_{exp}}$$
$$x_3^{exp} = \frac{1}{2\lambda_{exp}}(1+\lambda_{exp}-\sqrt{\lambda_{exp}^2-2\lambda_{exp}-3})$$
$$x_4^{exp} = \frac{1}{2\lambda_{exp}}(1+\lambda_{exp}+\sqrt{\lambda_{exp}^2-2\lambda_{exp}-3})$$

From these coordinates, the values x_1^{exp} and x_{exp}^2 are the experimental fixed points with period $p = 1$, while the values x_{exp}^3 and x_{exp}^4 are the experimental fixed points with period $p = 2$.

Stability of fixed points: Let us characterize the stability of the period-2 orbit $(\boldsymbol{p}_1, \boldsymbol{p}_2)$. According to the theory, these points have the same type of stability, and therefore the characterization of the stability of only one point allows us to characterize the whole trajectory. By applying the chain rule $DF^2(\boldsymbol{p}_1) = DF(\boldsymbol{p}_2)DF(\boldsymbol{p}_1)$ it can be proven that the Jacobian $DF^2(\boldsymbol{p}_1)$ is equal to:

$$DF^2(\boldsymbol{p}_1) = \begin{pmatrix} -\lambda_{exp}^2+2\lambda_{exp}+4 & \lambda_{exp}(\lambda_{exp}-2)-\lambda_{thr}(\lambda_{thr}-2) \\ 0 & -\lambda_{thr}^2+2\lambda_{thr}+4 \end{pmatrix} \tag{4}$$

with eigenvalues $\mu_1 = -\lambda_{thr}^2 - 2\lambda_{thr} + 4$ and $\mu_2 = -\lambda_{exp}^2 - 2\lambda_{exp} + 4$. Therefore, the regions of stability of the period-2 orbit (as they are calculated via the inequalities $-1 < -\lambda_{thr}^2 - 2\lambda_{thr} + 4 < 1$ and $-1 < -\lambda_{exp}^2 - 2\lambda_{exp} + 4 < 1$) are the regions $3 < \lambda_{thr} < 1+\sqrt{6}$ and $3 < \lambda_{exp} < 1+\sqrt{6}$ in complete analogy with the original logistic map model.

3 Periodicity of the Error Time Series

Let us consider the time series $\boldsymbol{p}_i = (e_i, x_i)$, x_i^{thr} and e_i (i=1,2,...,N). The periodicity of these time series it can be described by the next two theorems:

Theorem 1. *If the times series $\{x_1, x_2, \dots, x_k\}$ is periodic with a period value equal to $p = k$, then, the time series $\{\boldsymbol{p}_1, \boldsymbol{p}_2, \dots, \boldsymbol{p}_k\}$ is also periodic, and it is characterized by the same period value.*

PROOF: This theorem can be proven by using the induction method. (a) In the first step we have to prove that if the property $x_n^{thr} = f_2(e_n, x_n^{thr})$ holds, then the equation $\boldsymbol{p} = \boldsymbol{F}(\boldsymbol{p})$ also holds. This property has been proven in the section associated with fixed points with period $p = 1$: the theoretical trajectory as well as the trajectory $\boldsymbol{p}_i$ (i=1,2,...,N) were found to have the same period and the same stability and instability regions. (b) In the next step we accept that the property holds for period $p = k$, i.e. if the relation $x = f_2^k(e_n, x_n^{thr})$ is true, then the relation $\boldsymbol{p} = \boldsymbol{F}^k(\boldsymbol{p})$, is also true (c) finally we have to prove that the property holds for $p = k+1$, too. This is actually true, since $\boldsymbol{F}^{k+1}(\boldsymbol{p}) = \boldsymbol{F}(\boldsymbol{F}^k(\boldsymbol{p})) = \boldsymbol{F}(\boldsymbol{p}) = \boldsymbol{p}$. Therefore, the time series $\{x_1, x_2, \dots, x_k\}$ and $\{\boldsymbol{p}_1, \boldsymbol{p}_2, \dots, \boldsymbol{p}_k\}$ are characterized by the same periodicity, regarding the various regions of the values of the λ parameter.

Theorem 2. *If the time series $\{\boldsymbol{p}_1, \dots, \boldsymbol{p}_k\}$ and $\{x_1, \dots, x_k\}$ are periodic with period $p = k$ (according to the previous theorem), then, the error time series $\{e_1, \dots, e_k\}$, is also periodic, with period $p = k$.*

PROOF: If the time series $\{\boldsymbol{p}_1, \boldsymbol{p}_2, \dots, \boldsymbol{p}_k\}$ is periodic with period $p = k$, then, the chain rule $DF^k(\boldsymbol{p}_1) = DF(\boldsymbol{p}_k)DF(\boldsymbol{p}_{k-1})\dots DF(\boldsymbol{p}_1)$ can be applied. Regarding the Jacobian $DF(\boldsymbol{p}_i)$, it can be simplified furthermore, since the parameter x_{n+1} is independent of the parameter e_n, and therefore, the equation $(\partial f_2(e_n, x_n^{thr}))/(\partial e_n)$=0 holds. So, if we use the matrix equation:

$$\begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix} \dots \begin{pmatrix} a_k & b_k \\ 0 & c_k \end{pmatrix} = \begin{pmatrix} \prod_{i=1}^{k} a_i & Expr \\ 0 & \prod_{i=1}^{k} c_i \end{pmatrix} \tag{5}$$

(where Expr is an algebraic expression derived from the multiplication of the k matrices) the Jacobian $DF^k(\boldsymbol{p}_1)$ can be written in the form:

$$DF^k(\boldsymbol{p}_1) = \begin{pmatrix} (\frac{\partial^k f_1}{\partial e^k})_{\boldsymbol{p}_i} & (\frac{\partial^k f_1}{\partial x^k})_{\boldsymbol{p}_i} \\ 0 & (\frac{\partial^k f_2}{\partial x^k})_{\boldsymbol{p}_i} \end{pmatrix} = \begin{pmatrix} \prod_{i=1}^{k} (\frac{\partial f_1}{\partial e})_{\boldsymbol{p}_i} & A \\ 0 & \prod_{i=1}^{k} (\frac{\partial f_2}{\partial x})_{\boldsymbol{p}_i} \end{pmatrix} \tag{6}$$

and therefore

$$(\frac{\partial^k f_1}{\partial e^k})_{\boldsymbol{p}_1} = (\frac{\partial f_1}{\partial e})_{\boldsymbol{p}_k} (\frac{\partial f_1}{\partial e})_{\boldsymbol{p}_{k-1}} \dots (\frac{\partial f_1}{\partial e})_{\boldsymbol{p}_1} \tag{7}$$

This equation is identified as the chain rule associated with an one dimensional period-k orbit. Hence, we can claim that the if the time series $\boldsymbol{p}_i$ and x_i^{thr} $(i = 1, 2, \dots, N)$ are periodic with a period $p = k$, the same is true for the time series e_i $(i = 1, 2, \dots, N)$. If we apply this property for the period $p = 3$ then, according to Sharkovskii's theorem (period $p = 3$ implies chaos), it is clear, that in the chaotic regions of the time series $\boldsymbol{p}_i$ and x_i^{thr} $(i = 1, 2, \ldots, N)$, the time series e_i $(i = 1, 2, \ldots, N)$ is also chaotic. Therefore, if we record the error time series for various values of the λ parameter, the stable points of those time series will form a bifurcation diagram, similar to the diagram formed by the fixed points of the theoretical model of the logistic map.

4 Calculation of the Minimum Simulation Error

To calculate the minimum simulation error between the theoretical and the neural based time series samples, we have to identify the critical points of the function $f_1(e_n, x_n^{thr})$, whose coordinates must satisfy the system of equations

$$\frac{\partial f_1(e_n, x_n^{thr})}{\partial e_n} = +2\lambda_{exp} e_n + \lambda_{exp}(1 - 2x_n^{thr}) = 0 \tag{8}$$

$$\frac{\partial f_1(e_n, x_n^{thr})}{\partial x_n^{thr}} = -2\lambda_{exp} e_n + (\lambda_{thr} - \lambda_{exp})(1 - 2x_n^{thr}) = 0 \tag{9}$$

The mathematical analysis associated with this task, leads to an error value equal to $e_{min} = (\lambda_{thr} - \lambda_{exp})/4$. Therefore, the minimum simulation error value is equal to the difference between the maximum theoretical and the maximum experimental value of the time series of the dynamical system under consideration.

5 Experimental Results - Conclusions

The validity of theory presented in the previous sections was verified experimentally through the construction of neural models for the logistic map, for parameter values associated with the region of convergence as well as the periodic and the chaotic regions. After the training of the neural network, it was used to generate the experimental time series. In the next step this time series was compared to the theoretical one, and the time series of the simulation error between these two series was recorded. Then the value of the parameter λ_{exp} was estimated (via equation (1)) and compared against the corresponding value of the theoretical model λ_{thr}. The region that gave the best results was the converging region, while the coordinates of the experimental periodic points were proven very sensitive to the simulation accuracy. In the chaotic region the validity test for this theory was based on the calculation of the Luapunov exponent for the two dimensional trajectory (e_i, x_i) of the analytical model [8], as well as the simulation error time series produced by the neural network (in the last case the calculation of the Lyapunov exponent was performed by the Wolf's algorithm [9]). Based on these, the main conclusions of this research are the following:

- By using the assumption that the function simulated by the neural network is a second degree polynomial, it is proven that this function is actually the logistic equation with a value for the λ parameter, λ_{exp}, different than the parameter value of the theoretical model, λ_{thr}.

- The experimental fixed points can be calculated by applying the same analytical equations as in the theoretical case. However, the parameter value that has to be used in this calculation is not the λ_{thr}, but the λ_{exp}. This fact was proven for fixed points with period $p = 1$ and $p = 2$, but the generalization of this conclusion for any period value is straightforward.
- The time series (e_i, x_i) of the two dimensional theoretical model is characterized by the same periodicity with the time series $\{x_i^{thr}\}$ of the original logistic map model for the various values of the λ parameter. The same fact is true for the time series $\{e_i\}$ of the simulation error between the theoretical and the experimental system time series.
- By using the previous conclusion, we can say that if the stable fixed points of the error time series are recorded for various values of the λ parameter, they form a bifurcation diagram similar to it, created by the stable fixed point of the original logistic map model.
- It is proven that the minimum simulation error between the theoretical and the experimental time series is equal to $e_{min} = (\lambda_{thr} - \lambda_{exp})/4$, and therefore, equal to the difference between the maximum theoretical and the maximum experimental value of the system time series.
- The Lyapunov exponent of the time series $\{e_i\}$, $\{x_i\}$ and $\{(e_i, x_i)\}$, is characterized by the same qualitative behavior for the various values of the λ parameter of the system.

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